

# RATIONAL POSITIVE REAL APPROXIMATIONS FOR LQG OPTIMAL COMPENSATORS ARISING IN ACTIVE STABILIZATION OF FLEXIBLE STRUCTURES

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## SUMMARY

In this paper the approximation problem for a class of optimal compensators for flexible structures is considered. The particular case of a simply supported truss with an offset antenna is dealt with. The nonrational positive real optimal compensator transfer function is determined, and it is proposed that an approximation scheme based on a continued fraction expansion method be used. Comparison with the more popular modal expansion technique is performed in terms of stability margin and parameters sensitivity of the relative approximated closed loop transfer functions.

## INTRODUCTION

The problem of active stabilization of flexible structures with collocated sensors/actuators is addressed. In particular, the case of an offset antenna linked by a truss to the Shuttle body is considered. A general theory has been already established in [1] as optimal LQG problem for abstract wave equations. The results obtained were applied in [2] to design an optimal compensator for the antenna vibrations suppression after a slewing action, by modelling the truss as a uniform Bernoulli beam, simply supported at the Shuttle end, with rate-sensor/actuator collocated at the antenna end. The compensator transfer function was determined as nonrational, positive real function. This class of functions was shown to provide robust stabilizers for vibrating systems, even in the case of lumped parameter systems [3]. Nevertheless, rational approximation schemes are needed in order to instrument the compensator. A technique usually adopted is the modal expansion, and a typical controller realization can be found in [3] as a bank of filters, centered at the frequencies of the system undamped modes. Other methods can be borrowed from networks synthesis framework, where the rational approximation of positive real functions is a standard problem in telecommunications filters design. Standard references on these problems are [5], [6]. Despite a good approximation of some

characteristics of the frequency response (amplitude, real part, etc.) can be obtained, the positive real character of the approximating function is not guaranteed, as opposite the modal expansion does for the class of systems considered. This is crucial in our control problems since, as mentioned above, positive realness ensures the structure stabilization. Anyway, the main drawback is just the modal frequencies computation, obtained by solving a transcendental equation.

In this paper the compensator transfer function is explicitly computed in the vector case of yaw torsion plus roll bending deformation. It is shown to converge to a diagonal constant matrix as the control energy increases without bound. Rational positive real approximations are devised via a continued fraction expansion technique [7]. Approximations of any order can be easily derived, with coefficients straightforwardly related to the system parameters. Moreover, the positive real character is guaranteed. The performances of the approximated closed loop transfer function are evaluated. Comparison with the modal approximation method is performed in terms of stability margins and sensitivity to parameters variations.

## THE OPTIMAL COMPENSATOR DESIGN

We resume in this section the known results about the model and the LQG problem for the case of a simply supported uniform Bernoulli beam with an offset antenna, with rate-sensor/actuator collocated at the antenna end [2]. Actually, the particular case of roll bending deformation ( $x$  axis) plus yaw torsion ( $z$  axis) is considered. In the sequel  $u_\phi(t, s)$  denotes the  $x$  axis displacement and  $u_\psi(t, s)$  the angular displacement about the  $z$  axis;  $t$  and  $s \in (0, l)$  indicate the time variable and the space displacement along the beam axis, respectively. The starting point is the following state space model

$$M\ddot{x}(t) + Ax(t) + Bu(t) + BN_s(t) = 0 \quad (1.1)$$

$$v(t) = B^* \dot{x}(t) + N_o(t) \quad (1.2)$$

where

$x(t) \in$  Hilbert Space  $\mathcal{H}$

$M$  : linear bounded, self-adjoint positive definite operator on  $\mathcal{H}$  onto  $\mathcal{H}$ , with bounded inverse

$A$  : closed linear operator with domain dense in  $\mathcal{H}$  and range in  $\mathcal{H}$

$u(\cdot)$  : the control input  $\in E^n$ , Euclidean  $n$ -space

$B$  : linear mapping  $E^n \rightarrow \mathcal{H}$

$N_s(\cdot)$  : white Gaussian noise with spectral density  $dsI$ , where  $I$  is the identity operator of suitable dimension

$N_o(\cdot)$  : white Gaussian noise with spectral density  $d_oI$ , independent of  $N_s(\cdot)$

$v(\cdot)$  sensor data.

The definition of operators  $A$ ,  $M$  and  $B$ , as well as the definition of the space  $\mathcal{H}$ , embody the beam plus antenna dynamical model and the boundary conditions stated in [2]. For convenience we report these definitions:

$$x = \begin{bmatrix} u_\phi(\cdot) \\ u_\psi(\cdot) \\ u_\phi(l) \\ u'_\phi(l) \\ u_\psi(l) \end{bmatrix}; \quad Ax = \begin{bmatrix} A_f f \\ A_b f \end{bmatrix} = \begin{bmatrix} EI_\phi u_\phi''''(\cdot) \\ -GI_\psi u_\psi''(\cdot) \\ -EI_\phi u_\phi'''(l) \\ EI_\phi u_\phi''(l) \\ GI_\psi u_\psi'(l) \end{bmatrix}; \quad Bu = \begin{bmatrix} 0 \\ u(\cdot) \end{bmatrix} \quad Mx = \begin{bmatrix} M_f f \\ M_b b \end{bmatrix}$$

$$M_f f = \begin{bmatrix} \rho a u_\phi \\ \rho I_\psi u_\psi \end{bmatrix}; \quad M_b = \begin{bmatrix} m & 0 & m r_x \\ 0 & & \\ m r_x & I_M & \end{bmatrix},$$

where  $\rho$  is the beam mass density,  $a$  the cross sectional area,  $l$  the beam length,  $EI_\phi$ ,  $GI_\psi$  are the beam flexural and torsional rigidity respectively,  $m$  is the antenna mass and  $r_x$  is the antenna c.o.g displacement,  $I_M$  is the  $2 \times 2$  relevant moment of inertia matrix of the whole structure.

We consider the problem of stabilizing the antenna after a slewing action has occurred, by determining the control  $u(\cdot)$  that minimizes the time average

$$\lim_{T \rightarrow \infty} \frac{1}{T} \left( \int_0^T \|B^* \dot{x}(t)\|^2 dt + \lambda \int_0^T \|u(t)\|^2 dt \right), \quad \lambda > 0.$$

If  $(A, B)$  is controllable in [2] it is shown that the optimal compensator transfer function is given by

$$\Psi(\mu) = \alpha \mu B^* (\mu^2 M + A + \gamma \mu B B^*)^{-1} B \quad (1.3)$$

where

$$\alpha = \sqrt{d_s/d_o \lambda}, \quad \gamma = \sqrt{d_s/d_o} + \frac{1}{\lambda}.$$

$\Psi(\mu)$  is shown to be a positive real function, thus defining a robust controller [3].

## THE APPROXIMATION SCHEME

*The compensator transfer function explicit determination.* In order to compute explicitly the compensator transfer function from eqn. (1.3) consider the following expression

$$(\mu^2 M + A + \mu \gamma B B^*)x = Bv \quad (2.1)$$

which, by taking into account the definitions given in the previous section, can be split in the following two relationships

$$\mu^2 M_f f + A_f f = 0 \quad (2.2)$$

$$\mu^2 M_b b + A_b f + \mu \gamma b = v. \quad (2.3)$$

By recalling that  $f = [u_\phi(\cdot) u_\psi(\cdot)]^*$ , (2.2) is solved with the clamped end conditions  $u_\phi(0) = u'_\phi(0) = u_\psi(0) = 0$ , obtaining

$$\begin{aligned} u_\phi(s) &= c_1 \phi_1(s) + c_2 \phi_2(s) \\ u_\psi(s) &= c_3 \sinh(k_\psi s) \quad s \in [0, l), \end{aligned}$$

where

$$\begin{aligned} \phi_1(s) &= \sinh(k_\phi s) - \sin(k_\phi s) & \phi_2(s) &= \cosh(k_\phi s) - \cos(k_\phi s) \\ k_\phi &= (\rho a / (EI_\phi))^{1/4} \sqrt{i\mu} & k_\psi &= \sqrt{\rho/G} \mu. \end{aligned}$$

For the constants  $c_1, c_2$  the following formulas hold

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \phi_1(l) & \phi(l) \\ \phi'_1(l) & \phi'_2(l) \end{bmatrix}^{-1} \begin{bmatrix} \phi_1(l) \\ \phi_2(l) \end{bmatrix}; \quad c_3 = \frac{u_\psi(l)}{\sinh(k_\psi l)},$$

moreover

$$u_{\phi}^{(n)}(\cdot) = [\phi_1^{(n)}(\cdot) \quad \phi_2^{(n)}(\cdot)] \begin{bmatrix} \phi_1(l) & \phi(l) \\ \phi_1'(l) & \phi_2'(l) \end{bmatrix}^{-1} \begin{bmatrix} \phi_1(l) \\ \phi_2(l) \end{bmatrix}, \quad n = 0, 1, 2, \dots$$

Now it can be shown that  $A_b f$  can be actually expressed as  $T(\mu)b$  [4] so that (2.1) becomes

$$(\mu M_b + T(\mu) + \mu \gamma I)b = v$$

and the compensator transfer function is defined as

$$\Psi(\mu) = \alpha \mu (\mu^2 M_b + T(\mu) + \mu \gamma I)^{-1}, \quad (2.4)$$

where  $I$  is the identity operator. Recalling the definition of  $\alpha$  and  $\gamma$ , we see that

$$\lim_{\lambda \rightarrow 0} \Psi(\mu) = \sqrt{d_s/d_o} I$$

so that we have the “direct connection” if the control energy increases without bound.

We list below the non zero entries of the  $3 \times 3$  matrix  $T(\mu)$

$$T_{11}(\mu) = -EI_{\phi} k_{\phi}^3 \frac{\sinh(k_{\phi} l) \cos(k_{\phi} l) + \sin(k_{\phi} l) \cosh(k_{\phi} l)}{\Delta}$$

$$T_{12}(\mu) = T_{21}(\mu) = EI_{\phi} k_{\phi}^2 \frac{\sinh(k_{\phi} l) \sin(k_{\phi} l)}{\Delta}$$

$$T_{22}(\mu) = EI_{\phi} k_{\phi} \frac{\sinh(k_{\phi} l) \cos(k_{\phi} l) - \sin(k_{\phi} l) \cosh(k_{\phi} l)}{\Delta}$$

$$T_{33}(\mu) = GI_{\psi} k_{\psi} \coth(k_{\psi} l)$$

$$\Delta = -1 + \cosh(k_{\phi} l) \cos(k_{\phi} l).$$

$T(\mu)$  is a meromorphic function; i.e. it is analytic in all the plane but in a countable set of points, where it has polar singularities [8].

*The continued fraction expansion approximation.* In order to approximate (2.4) note that if a matrix is positive real so is its inverse; consequently it is of great simplification to work on

$$\frac{1}{\mu}(\mu^2 M_b + T(\mu) + \gamma \mu I). \quad (3.1)$$

Next, since for  $\mu = i\omega$  the function  $T(\mu)$  is real, we see that the real part of (2.5) does not depend on  $T(\mu)$ , so that whatever approximation we device, a positive real approximate function is obtained. Thus let us concentrate on  $T(\mu)$ .

The approximation scheme proposed consists of a continued fraction expansion of a meromorphic function  $f(z)$ , i.e.

$$f(z) = r_0(z) + \frac{1}{r_2(z) + \frac{1}{r_3(z) + \frac{1}{r_4(z) + \dots}}} \quad (3.2)$$

where the  $r_i(z)$ ,  $i = 0, 1, 2, \dots$  are rational functions of finite degree suitably defined, and are called “convergents” of the continued fraction. The meaning of (3.1) is the following [7]: denoting  $f_n(z)$  the function obtained by considering  $n$  convergents on the r.h.s we have

$$\lim_{n \rightarrow \infty} f_n(z) = f(z)$$

for every value of  $z$  in the complex plane.

The functions  $r_i(z)$  can be determined according to the following algorithm.

*step 1.* Consider the Laurent expansion about the origin of  $f(z)$  (see e.g. any standard book on complex analysis or passive networks synthesis)

$$f(z) = \sum_{k=0}^{n_0} a_{-k} z^{-k} + \sum_{k=1}^{\infty} a_k z^k.$$

The finite sum is called “singular part” of  $f(z)$  at the origin,  $n_0$  is the multiplicity of the pole at the origin and set  $r_0(z) = \sum_{k=0}^{n_0} a_{-k} z^{-k}$ ; it can eventually be a constant ( $n_0 = 0$ ); i.e. the function is regular at  $z = 0$ .

*step 2.* Compute

$$f_1(z) = f(z) - r_0(z),$$

then  $f_1(z)$  has the same poles as  $f(z)$  but the pole at  $z = 0$ . Moreover it holds

$$f(z) = r_0(z) + f_1(z) \quad (i)$$

$$\lim_{z \rightarrow 0} f_1(z) = 0. \quad (ii)$$

Note that (i) is not a local expansion but an exact representation of  $f(z)$  with its singular behaviour at  $z = 0$  explicitated.

*step 3.* Consider

$$f_2(z) = 1/f_1(z);$$

it has a pole in the origin according to (ii), then use the argument of steps 1 and 2 to determine  $f_2(z) = r_1(z) + f_3(z)$  with  $r_1(z) = \sum_{k=0}^{n_1} b_{-k} z^{-k}$  and  $n_1$  is the multiplicity of the pole at  $z = 0$  of  $f_2(z)$ . Since  $f_1(z) = 1/f_2(z)$  we have

$$f(z) = r_0(z) + \frac{1}{r_1(z) + f_3(z)}.$$

It is clear at this point how to obtain expansion (3.1) by repeating step 3 and determining recursively all the convergents  $r_k(z)$ ,  $k = 0, 1, 2, \dots$ . For completeness let us see how to easily obtain the coefficients of the singular part of the Laurent expansion at the origin of a meromorphic function  $f(z)$

$$a_{-n} = \lim_{z \rightarrow 0} z^n f(z)$$

$$a_{-k} = \lim_{z \rightarrow 0} z^k \left( f(z) - \sum_{i=1}^{n-k} a_{-n-1+i} z^{-n-1+i} \right).$$

The algorithm described can be applied to each entry of  $T(\mu)$  obtaining an expansion of the following type

$$R^{i,j}(\mu) = k_0^{i,j} + \frac{1}{\frac{k_{11}^{i,j}}{\mu^4} + k_{10}^{i,j} + \frac{1}{\frac{k_{21}^{i,j}}{\mu^4} + k_{20}^{i,j} + \frac{1}{\frac{k_{31}^{i,j}}{\mu^4} + k_{30} + \frac{1}{\dots}}}}, \quad (3.3)$$

and consequently the  $n$ -th order approximation  $T_n(\mu)$  of  $T(\mu)$  is obtained as

$$T_n(\mu) = \begin{bmatrix} \frac{EI_\phi}{l^3} R_n^{1,1}(\mu) & \frac{EI_\phi}{l^2} R_n^{1,2}(\mu) & 0 \\ \frac{EI_\phi}{l^2} R_n^{2,1}(\mu) & \frac{EI_\phi}{l} R_n^{2,2}(\mu) & 0 \\ 0 & 0 & \frac{GI_\psi}{l} R_n^{3,3}(\mu) \end{bmatrix}$$

Here  $R_n^{i,j}(\mu)$  is obtained by (3.3) including  $n$  convergents in the continued fraction. This number needs not be the same for all the entries, and in this case the order for  $T_n(\mu)$  is determined by the highest value of  $n$  used.

In the following table are reported the coefficients of the continued fraction expansion up to the 4-th order of the entries of  $T(\mu)$ , for a particular choice of the system parameters (see [9]).

Table 1. Coefficients of the Continued Fraction Expansion of  $T(\mu)$  entries.

	$R^{1,1}$	$R^{1,2}$	$R^{2,2}$	$R^{3,3}$
$k_0$	12	-6	4	1
$k_{11}$	-2.7	19.1	-105	3
$k_{10}$	.0026	-.028	.18	.2
$k_{21}$	411815.38	-74697.1	-10.2	-175
$k_{20}$	-451.03	50.57	21610.7	-7.7
$k_{31}$	94.35	714.4	-3434.97	35.64
$k_{30}$	.0185	-.104	.4	.61
$k_{41}$	5501900	-921286.76	244973.48	-782.41
$k_{40}$	321.53	44.58	-10.15	-7.08

We stress that the coefficients of the approximations are easily obtained just by computing limits in the origin of suitably defined functions, and are simple combinations of the system parameters which appear in the coefficients of the function to be approximated. As a result we have a procedure which is numerically robust



since no solution of transcendental equations are required, no derivatives determination, no inner products computations, no variable transformations as usually occurs in the most popular approximation methods. This straightforward relation to the system parameters determines also a low sensitivity to parameters variation of the approximated closed loop transfer functions, as we will show next.

For convenience, we report briefly the modal expansion technique (see [3]). Let  $\omega_k$ ,  $k = 1, 2, \dots$ , the system undamped modes, and  $\psi_k$ ,  $k = 1, 2, \dots$ , the  $M$ -orthogonal eigenvectors, normalized as  $[M\psi_k, \psi_k] = 1$ . The following compensator transfer function modal approximation can be devised

$$\sqrt{\frac{d_s}{\lambda d_0}} \sum_{k=1}^{\infty} \frac{\mu b_k b_k^*}{\mu^2 + \omega_k^2 + \mu \gamma b_{kk}}, \quad \text{Re} \mu \geq 0,$$

where

$$b_k = B^* \psi_k, \quad b_{kk} = [b_k, b_k].$$

Thus we have a bank of band pass filters centered at the undamped modes.

#### CLOSED LOOP TRANSFER FUNCTION PERFORMANCES EVALUATION

In this section we compare the performances of the approximation method proposed with respect to the modal expansion technique, widely used in this field. In particular the stability margins of the approximate closed loop transfer function are considered and the sensitivity of this performance index toward system parameter variation is evaluated.

As it is well known the stability properties mentioned can be derived by examining the frequency behaviour of the following function

$$S(\omega) = \det(I + P(\omega)\Psi(\omega)),$$

where  $P(\omega)$  is the system transfer function, in our case defined as

$$P(\omega) = (T(\omega) - \omega^2 M_b)^{-1}.$$

Actually, we are more interested in the sensitivity of stability margins with respect to parameters variations. In Fig. 1 amplitude and phase plots of the diagonal

entries of  $F(\omega) = P(\omega)\Psi(\omega)$  are reported for a second order continued fraction approximation of  $\Psi(\omega)$ . In Fig. 2 are reported the corresponding plots for a modal approximation based on two band pass filters centered on the first two modes. Particularly interesting are the phase plots showing that in the first case we obtain a higher phase margin, practically equal to  $\pi/2$  for all frequencies. A similar result holds for the other entries of  $F(\omega)$ , thus giving a description of the stability features achieved in both approximation schemes, avoiding to get through the complexity of the function  $S(\omega)$ .

Moreover, in Fig. 3 are reported the plots of the sensitivity of the phase functions considered, with respect to the variation of the parameter  $\theta = \frac{\rho a}{EI_\phi}$ . Here we note that the continued fraction approximation shows a better performance in terms of robustness than the modal expansion.

## CONCLUSIONS

The continued fraction method proposed allows to approximate any meromorphic function by operating simple computations on the coefficients, i.e. the determinations of the limit in the origin of suitable functions derived by the assigned one. This results in a good performance of the approximation in terms of stability margins and robustness of the approximate closed loop transfer function. This feature is highlighted by comparing the mentioned characteristics with the analogous one obtained by using the more popular modal approximation scheme.

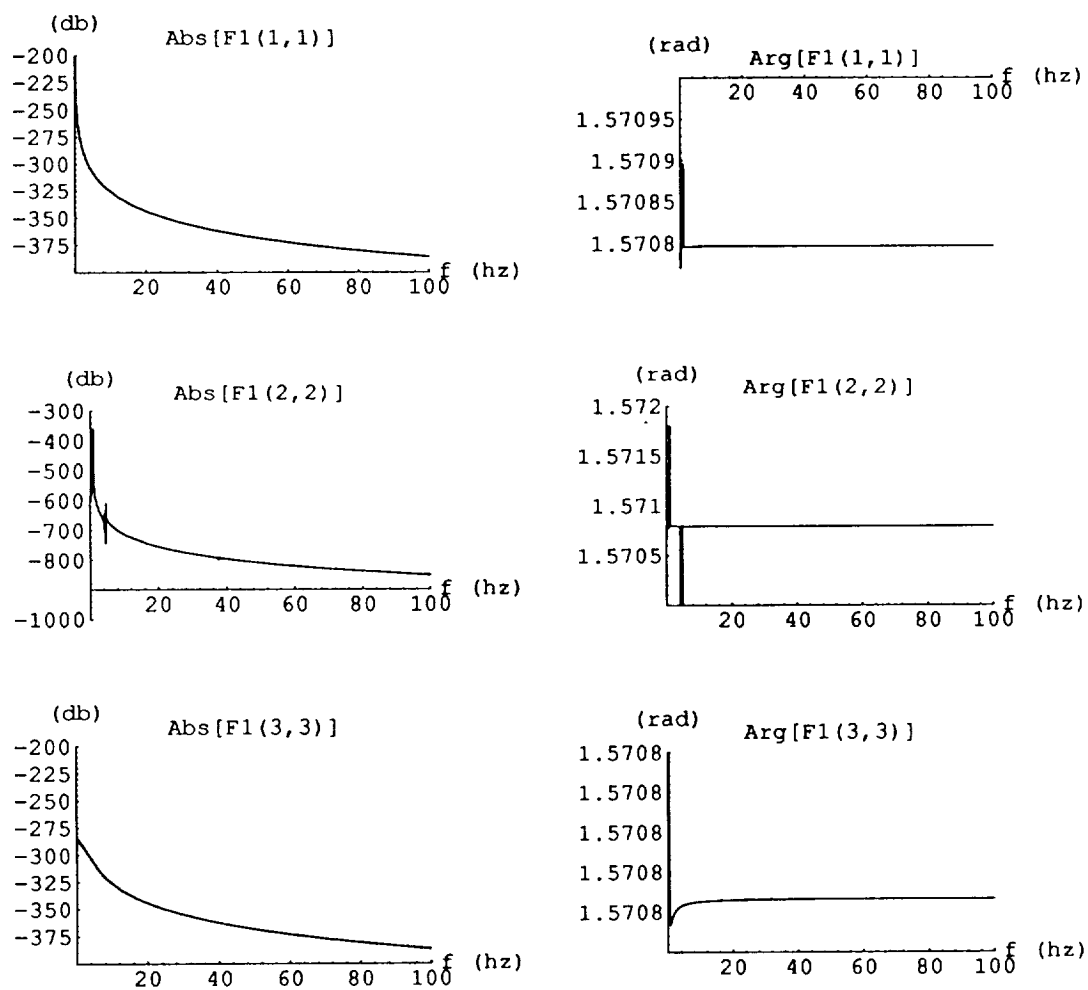


Fig. 1 Continued Fraction Approximation.

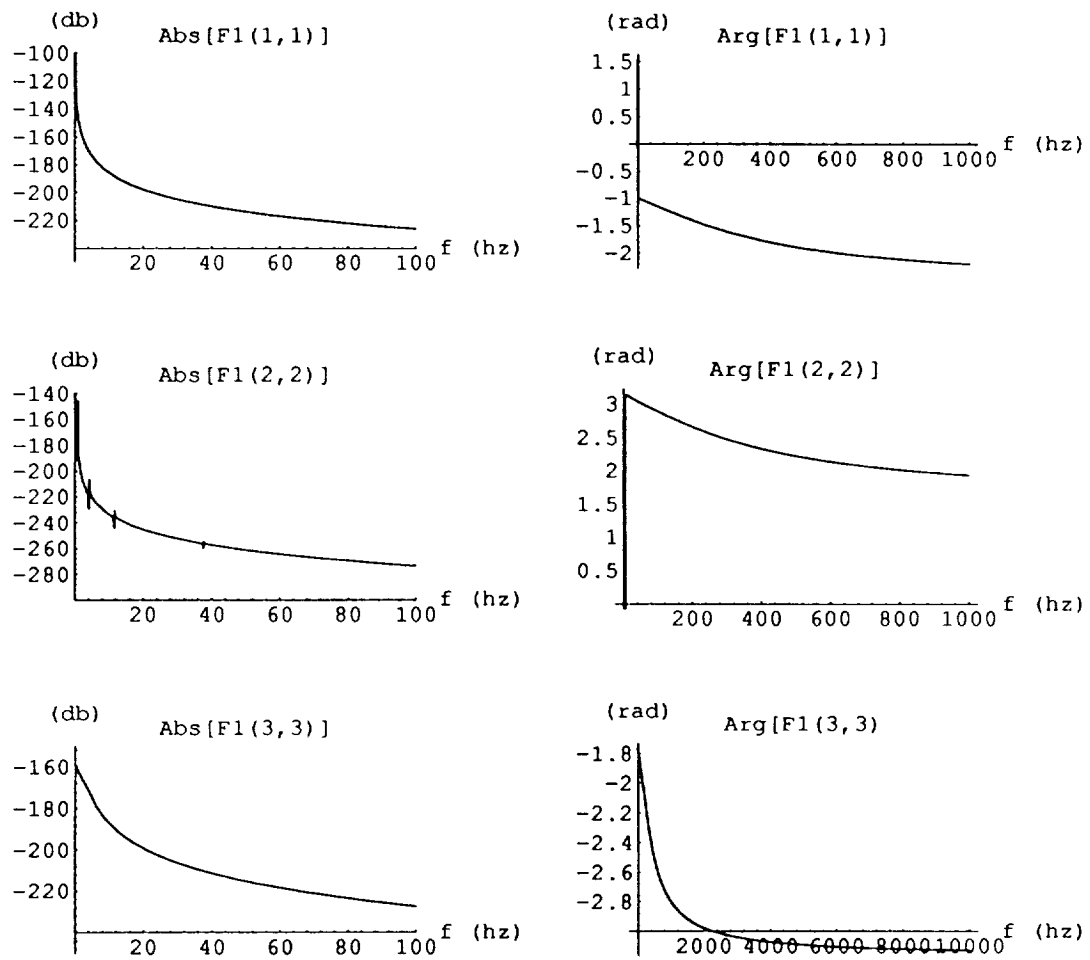
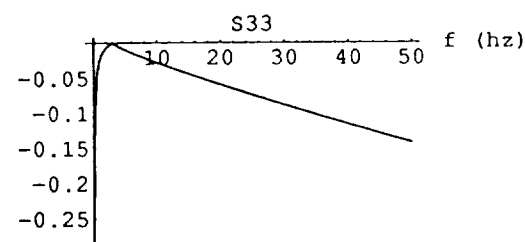
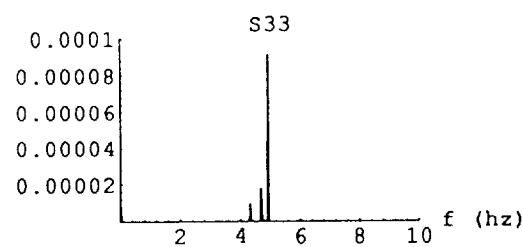
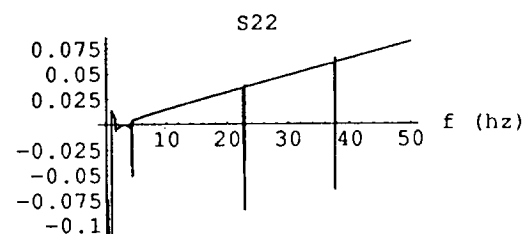
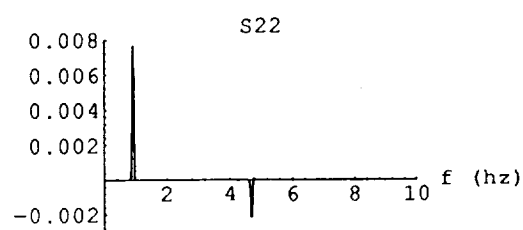
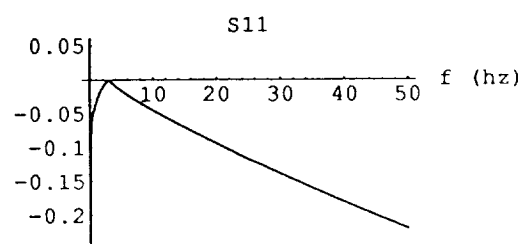
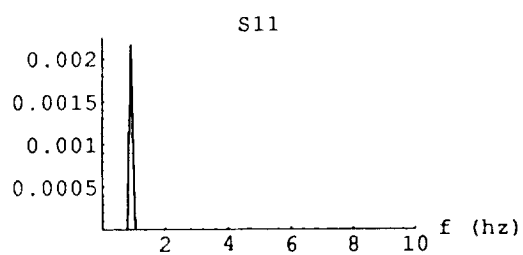


Fig. 2 Modal Expansion Approximation.



Continued Fraction Approximation

Modal Expansion Approximation

Fig. 3 Sensitivity to Parameter  $\theta = \frac{\rho a}{EI_\phi}$

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